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The Differential Transform Method for Solving Volterra's Population Model

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In this article, Differential transform method is presented for solving Volterra's population model for population growth of a species in a closed system. This model is a nonlinear integro-differential where the integral term represents the effect of toxin. This powerful method catches the exact solution. First Volterra's population model has been converted to power series by one-dimensional differential transformation. Thus we obtained numerical solution Volterra's population model.

Introduction

This note deals with a mathematical model of the accumlated effect of toxins on a population living in a closed system. Scudo indicates in his review that volterra proposed this model for a population $q(k)$ of identical individuals wich exhibits crowding and sensitivity to "total metabolism":

$$\frac{dq}{dt} = aq - bq^2 - cq\int_0^t q(x)\,dx, \qquad q(0) = q_0, \tag{1.1}$$

Where $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient and $c > 0$ is the toxicity coefficient. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long term. q_0 is the initial population

and $q = q(t)$ denotes the population at time t. This model is a first-order integro-ordinary

differential equation where the term $cq\int_0^t q(x)\,dx$, represents the effect of toxin accumulation on the species. We apply scale time and population by introducing the nondimensional variables

$$\lambda = \frac{tc}{b}, \qquad u = \frac{qb}{a},$$

to obtain the nondimensional problem:

$$k\frac{du}{d\lambda} = u - u^2 - u\int_0^\lambda u(x)\,dx \qquad u(0) = u_0, \tag{1.2}$$

Where $u(\lambda)$ is the scaled population of identical individuals at time λ and is a prescribed non- dimensional parameter. The model is characterized by the nonlinear Volterra integro-differential equation

$$k\frac{du}{d\lambda}=u-u^{2}-u\int_{0}^{\lambda}u(x)\,dx$$
$$u(0)=0.1, \tag{1.3}$$

and k is a prescribed parameter. The non dimensional parameter is $k=\frac{c}{ba}$.

In recent years, numerous works have been focusing on the development of more advanced and efficient methods for Volterra's population model such as singular perturbation method [5]. For example homotopy perturbation method [6], quasilinerization approach method [4] and Adomian decomposition method and Sinc-Galerkin method compared for the solution of some mathematical population growth models [3]. In [1], the series solution method and The decomposition method for Volterra's population model is considered. In 1986, Zhou [2] first introduced the differential transform method (DTM) in solving linear and nonlinear initial value problems in the electrical circuit analysis. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. the results of applying differential transformation method to the Volterra's population model will be presented.

One-Dimensional Differential Transform

Differential transform of function $y(x)$ is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^{k}y(x)}{dx^{k}}\right]_{x=0}, \tag{2.1}$$

In equation (2.1), $y(x)$ is the original function and $Y(k)$ is the transformed function, which is called the T-function. Differential inverse transform of $Y(k)$ is defined as

$$y(x)=\sum_{k=0}^{\infty}x^{k}Y(k), \tag{2.2}$$

From equation (2.1) and (2.2), we obtain

$$y(x)=\sum_{k=0}^{\infty}\frac{x^{k}}{k!}\left[\frac{d^{k}y(x)}{dx^{k}}\right]_{x=0}, \tag{2.3}$$

Equation (2.3) implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically.

However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original functions. In this study we use the lower case letter to represent the original function and upper case letter represent the transformed function.

From the definitions of equations (1.2) and (2.2), it is easily proven that the transformed functions comply with the basic mathematics operations shown in Table 1.

In actual applications, the function $y(x)$ is expressed by a finite series and equation (2.2) can be written as:

$$y(x)=\sum_{k=0}^{m}x^{k}Y(k), \tag{2.4}$$

Equation (2.3) implies that $\sum_{k=m+1}^{\infty}x^{k}Y(k)$ is negligibly small. In fact, m is decided by the convergence of natural frequency in this study.

Table 1. *The fundamental operations of one-dimensional differential transform method.*

Original function	Transformed function
$y(x)=u(x)\pm v(x)$	$Y(k)=U(k)\pm V(k)$
$y(x)=cw(x)$	$Y(k)=cW(k)$
$y(x)=dw/dx$	$Y(k)=(k+1)W(k+1)$
$y(x)=d^{j}w/dx^{j}$	$Y(k)=(k+1)(k+2)\cdots(k+j)W(k+j)$
$y(x)=u(x)v(x)$	$Y(k)=\sum_{r=0}^{k}U(r)V(k-r)$

Theorem 1. if $y(x)=\exp(x)$ then $W(k)=\frac{1}{k!}$.

Proof: By using equation (2.1), we have

$$W(k)=\frac{1}{k!}\frac{\partial(e^{x})}{\partial x^{k}}\bigg|_{t=0}=\frac{1}{k!}e^{x}\Big|_{x=0}=\frac{1}{k!}.$$

Theorem 2. if $y(x)=\frac{\partial^{m}u(x)}{\partial x^{m}}$ then $Y(k)=\frac{(k+m)!}{k!}U(k+m)$.

Proof: By using equation (2.1), we have

$$Y(k)=\frac{1}{k!}\frac{\partial^{k}}{\partial x^{k}}(\frac{\partial^{m}u(x)}{\partial x^{m}})\bigg|_{t=0}=\frac{1}{k!}\frac{\partial^{m+k}u(x)}{\partial x^{m+k}}\bigg|_{t=0}=\frac{(k+m)!}{k!}U(k+m).$$

Theorem 3. if $y(x)=x^{m}$ then $Y(k)=\delta(k-m)=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}$

Proof: By using equation (2.1), we have

$$W(k)=\frac{1}{k!}\frac{\partial^{k}(x^{m})}{\partial x^{k}}\bigg|_{t=0}=\begin{cases}\frac{1}{k!}\frac{\partial^{k}(x^{k})}{\partial x^{k}}=\frac{k!}{k!}=1 & k=m\\ \frac{1}{k!}\frac{\partial^{k}(x^{m})}{\partial x^{k}}=0 & k\neq m\end{cases}$$

$$W(k)=\delta(k-m)=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}.$$

Theorem 4. if $w(x)=\sin(wx+\alpha)$ then $.W(k)=\frac{w^{k}}{k!}\sin(\frac{k\pi}{2}+\alpha)$

Proof: By using equation (2.1), we have

$$k=1 \qquad W(1)=\frac{1}{1!}\frac{\partial\sin(wx+\alpha)}{\partial x}\bigg|_{x=0}=\frac{1}{1!}w\cos(wx+\alpha)\Big|_{x=0}$$

$$=\frac{1}{1!}w\sin((\frac{\pi}{2}+\alpha)+wx)\Big|_{x=0}=\frac{1}{1!}w\sin(\frac{\pi}{2}+\alpha),$$

$$k=2 \qquad W(2)=\frac{1}{2!}\frac{\partial^2 \sin(wx+\alpha)}{\partial t^2}\bigg|_{x=0} = \frac{1}{2!}w^2\cos(\frac{\pi}{2}+\alpha+wx)\big|_{x=0}$$

$$=\frac{1}{2!}w^2 sin(\frac{\pi}{2}+\left(\alpha+\frac{\pi}{2}\right)+wx)\bigg|_{x=0} = \frac{1}{2!}w^2\sin(\frac{2\pi}{2}+\alpha+wx)\big|_{x=0} = \frac{1}{2!}w^2\sin(\frac{2\pi}{2}+\alpha),$$

In the general form we have

$$k=n \qquad W(n)=\frac{1}{k!}\frac{\partial^n \sin(wx+\alpha)}{\partial x^n}\bigg|_{x=0} = \frac{w^n}{n!}\sin(\frac{n\pi}{2}+\alpha+wx)\bigg|_{x=0}$$

$$=\frac{w^n}{n!}\sin(\frac{n\pi}{2}+\alpha).$$

Theorem 5. if $w(x)=\cos(wx+\alpha)$ then $W(k)=\frac{w^k}{k!}\cos(\frac{k\pi}{2}+\alpha).$

Proof: By using equation (2.1), we have

$$k=1 \qquad W(1)=\frac{1}{1!}\frac{\partial \cos(wx+\alpha)}{\partial t}\bigg|_{x=0} = -\frac{1}{1!}w\sin(wx+\alpha)\big|_{x=0}$$

$$=\frac{1}{1!}w\cos((\frac{\pi}{2}+\alpha)+wx)\bigg|_{x=0} = \frac{1}{1!}w\cos(\frac{\pi}{2}+\alpha),$$

$$k=2 \qquad W(2)=\frac{1}{2!}\frac{\partial^2 \cos(wx+\alpha)}{\partial t^2}\bigg|_{x=0} = -\frac{1}{2!}w^2\sin(\frac{\pi}{2}+\alpha+wx)\big|_{x=0}$$

$$=\frac{1}{2!}w^2\left(\cos\frac{\pi}{2}+\left(\alpha+\frac{\pi}{2}\right)+wx\right)\bigg|_{x=x_0} == \frac{1}{2!}w^2\cos(\frac{2\pi}{2}+\alpha+wx)\big|_{x=0} = \frac{1}{2!}w^2\cos(\frac{2\pi}{2}+\alpha),$$

In the general form we have

$$k=n \qquad W(n)=\frac{1}{k!}\frac{\partial^n \sin(wx+\alpha)}{\partial x^n}\bigg|_{x=0} = \frac{w^n}{n!}\cos(\frac{n\pi}{2}+\alpha+wx)\bigg|_{x=0}$$

$$=\frac{w^n}{n!}\cos(\frac{n\pi}{2}+\alpha) = \frac{w^n}{n!}\cos(\frac{n\pi}{2}+\alpha).$$

Numerical Example

In this section, we use one dimensional differential transform method for solving the population growth model characterized by the nonlinear Volterra integro-differential equation

$$\frac{du}{dt} = 10u(t) - u^2(t) - 10u(t)\int_0^t u(x)\,dx, \tag{3.1}$$

With the initial condition

$$u(0)=0.1, \tag{3.2}$$

When takig the one dimensional differential transform of (3.1), we can obtian:

$$U(h+1)=\frac{1}{(h+1)}\left[10U(h)-10\sum_{s=0}^{h}U(s)U(h-s)-10\sum_{s=1}^{h}\frac{1}{s}U(h-s)U(s-1)\right], \tag{3.3}$$

From the initial condition (3.2), we can write

$$U(0)=0.1\,, \tag{3.4}$$

For each h substituting into Eq. (3.3) and by recursive method, the values $U(h)$ can be evaluated as follows:

$$\begin{aligned}&U(0)=0.1,\ U(1)=0.9,\ U(2)=3.55,\ U(3)=6.316666667,\ U(4)=-5.5375499998,\\ &U(5)=-63.74166667,\ U(6)=-156.12375,\ldots,\end{aligned} \tag{3.5}$$

by using the inverse transformation rule for one dimensional in Eq. (2.1), the following solution can be obtained:

$$\begin{aligned}u(t)&=\sum_{h=0}^{\infty}U(h)\ x^{h}=U(0)+U(1)t^{1}+U(2)t^{2}+U(3)t^{3}+U(4)t^{4}+U(5)t^{5}+U(6)t^{6}+\ldots,\\ &=0.1+0.9t+3.55t^{2}+6.316666667t^{3}-5.5375499998\ t^{4}-63.74166667t^{5}-156.12375t^{6}+O(t^{7})\ .\end{aligned} \tag{3.6}$$

In a complete agreement with the results previously obtained in the previous sections.

Conclusion

In this study, the differential transform method for the solution of the Volterra's population model is successfully expanded. It is observed that the method is robust. The method gives rapidly converging series solutions.The numerical results show that differential transform method is an accurate and reliable numerical technique for the solution of the Volterra's population model. ■

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References

[1] A. M. Wazwaz, Analytical approximations and Padé approximants for Volterra's population model, Applied Mathematics and Computation, 100 (1999) 13-25.

[2] J. K. Zhou, Huazhong University Press, Wuhan, China, 1986.

[3] K. Al-Khaled, Numerical Approximations for Population Growth Models. Appl. Math. Comput, 160 (2005) 865-873.

[4] K. Parand, M. Ghasemi ASEMI, S. Rezazadeh, A. Peiravi, A. Ghorbanpour, A. Tavakoli golpaygani, quasilinerization approach for solving Volterra's population model, Appl. Comput. Math., (2010)95-103.

[5] R. D. Small, Population growth in a closed system and Mathematical Modeling, in: Classroom Notes in Applied Mathematics, SIAM, Philadelphia, PA, (1989) 317-320.

[6] S. T. Mohyud-Din, A. Yıldırım, Y. Gūlkanat, Analytical solution of Volterra's population model, Journal of King Saud University 22 (2010) 247–250.

[7] Arikoglu A, Ozkol I. Solution of differential?difference equations by using differential transform method. Appl. Math. Comput, 181 (2006) 153–162.

[8] Arikoglu A, Ozkol I, Solution of fractional differential equations by using differential transform method. Chaos Soliton. Fract, 34 (2007) 1473–1481.

[9] E. Celik, Kh. Tabatabaei, Solving a class of volterra integral equation systems by the Differential Transform Method, 9 (2010) 1-5.

[10] V. S. Ertürk, Differential transformation method for solving differential equations of Lane-Emden type, Math. Comput. Appl, 12 (2007) 135-139.